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Moments of inertia and the shapes of Brownian paths

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Abstract. We compute the joint probability law of the principal moments of inertia of Brownian paths (open or closed), using constrained path integrals and random matrix theory. The case of two-dimensional paths is discussed in detail. In particular, we show that the ratio of the average values of the largest and smallest moments is equal to 4.99 (open paths) and 3.07 (closed paths). We also present results of numerical simulations, which include investigation of the relationships between the moments of inertia and the arithmetic area enclosed by a path.

The shape of a typical random walk is far from spherical, as was first noticed long ago [1]. The precise amount of this anisotropy is likely to be of importance in the hydrodynamics of dilute polymer fluids, especially for times less than the largest relaxation time of a single chain; in this regime, the relevant variable is the instantaneous shape of the molecule, rather than an average over all the orientations that the latter takes over a longer period of time [2, 3]. Indeed the velocity spectra of certain laminar polymeric flows exhibit fluctuations which are manifestly non-turbulent; it has been argued [4] that these velocity fluctuations can be accounted for if one drops the assumption of a spherical polymer configuration, replacing it with a prolate ellipsoidal ansatz. It is clear that the motion of an elongated molecule will be somewhat different from that of a spherical one, because the flow will make it flip around itself. As to the origin of this prolate shape, it cannot be attributed wholly to a dynamical effect, since in some of the flows considered in [4] no deformation of spherical molecules is expected. Hence non-sphericity must already be present at the static level (though the precise instantaneous shape of the molecule is of course affected by the flow, e.g. the ellipsoid can compress and stretch along its axis). Thus an obvious prerequisite of the phenomenological description of polymer flows is the knowledge of the average shape of a chain at rest.

In order to get information on the latter, a quantitative measure of 'shape' is needed. The best candidate is the inertial tensor T, or more precisely T up to the orientations of the chain, i.e. the set of its eigenvalues $\lambda_1, \ldots, \lambda_d$. However, the computation of their probability law $P(\lambda_1, \ldots, \lambda_d)$ is in general a difficult task. The problem is greatly simplified when one adopts the less ambitious approach of defining shape using only polynomial invariants of T, namely the symmetric polynomials of $\lambda_1, \ldots, \lambda_d$, the simplest example of which is of course $\mathrm{tr}(T) = R^2$, the square radius of gyration of the chain. A variety of results concerning the distributions of R^2 [5] and its orthogonal components [6–8] in fixed or random axes has been obtained in $d = 3$, and linear combinations of those components have been used to

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investigate the anisotropy of random walk chains [9]. Combining R^2 with other invariants of T, it is possible to improve on this and provide better characterizations of the shape of a walk, defining quantities known as asphericity, acylindricity, etc [10–12], the average values of which are more easily computed than those of the eigenvalues themselves. For instance, the average asphericity of a Brownian path is known in any dimension [11, 12] and may be compared with simulation results [9–15]; the ratios of the average values of the principal moments of inertia λ_i have been computed numerically in $d = 3$, for paths either open or closed [14, 15]. The more difficult problem of the self-avoiding path was tackled in [12] through renormalization group methods and the invariants computed in dimension $4 - \epsilon$; the excluded-volume effects have also been studied numerically [3,12–15]. As regards the λ_i themselves, their law has been expressed analytically in $d = 2$ for closed paths [16, 27], whereas in higher dimensions only numerical or asymptotic (in $1/d$) results have been obtained so far [17].

Here we address the issue of determining the probability law $P(\lambda_1, \ldots, \lambda_d)$ of the moments of inertia of a Brownian path, open or closed. Part of this law can easily be deduced from the fact that T is a random symmetric real matrix. We then obtain, using a path-integral representation, the whole law, from which we deduce the integrated laws $P(\lambda_+ - \lambda_-)$ ($d = 2$, λ_+ (λ_-) being the largest (smallest) eigenvalue of T) and $Q(R^2)$; these laws may be computed numerically and exact moments and asymptotic behaviours may be extracted. Furthermore, it turns out that in two dimensions the ratio $\langle\lambda_+\rangle/\langle\lambda_-\rangle$, which obviously constitutes a straightforward measure of anisotropy, may be computed exactly . All our results are compared with simulations obtained by generating (open or closed) random walks on a lattice. We also numerically examine the law of the arithmetic area enclosed by a path and argue that the latter is not unrelated to the distribution of the moments of inertia.

We first address the problem in an arbitrary number of dimensions d and consider the inertial tensor $T = (T_{ij}), 1 \leqslant i, j \leqslant d$ of an open Brownian path of length t (i.e. a continuous map $r : [0, t] \to R^d$). The probability distribution $P^{\circ}(T)$ can be written as a constrained path integral [18]:

$$P^{\circ}(T) = N \int \mathrm{d}^d r' \mathrm{d}^d r_{\mathrm{G}} \int_{r(0)=0}^{r(t)=r'} \mathcal{D}r \; \mathrm{e}^{-\int_0^t \dot{r}^2(\tau)\mathrm{d}\tau} \prod_{i \leqslant j} \delta(f_{ij}) \delta^d(g) \tag{1}$$

where N is a normalization constant, and

$$f_{ij} = \frac{1}{t} \int_0^t \mathrm{d}\tau (x_i(\tau) - x_{\mathrm{G}i})(x_j(\tau) - x_{\mathrm{G}j}) - T_{ij} \qquad g = \frac{1}{t} \int_0^t \mathrm{d}\tau \, r(\tau) - r_{\mathrm{G}} \tag{2}$$

where x_i and $x_{\mathrm{G}i}$ are the components of r and r_{G}; the diffusion constant is taken equal to $\frac{1}{2}$.

Using translational invariance we rewrite (1) in the more symmetric form

$$P^{\circ}(T) = N \int \mathrm{d}^d r_{\mathrm{A}} \mathrm{d}^d r_{\mathrm{B}} \int_{r(0)=r_A}^{r(t)=r_B} \mathcal{D}r \; \mathrm{e}^{-\int_0^t \dot{r}^2(\tau)\mathrm{d}\tau} \prod_{i \leqslant j} \delta(f_{ij}^{(0)}) \delta^d(g^{(0)}) \tag{3}$$

(the constraints $f_{ij}^{(0)}$ and $g^{(0)}$ are obtained by setting $r_{\mathrm{G}} = 0$ in (2)).

The inertial tensor T of the random path r is a random positive symmetric matrix; moreover, the rotational invariance of the distribution of the Brownian paths ensures that the matrices T make up an orthogonal ensemble. This ensemble is obviously not the so-called 'Gaussian orthogonal ensemble' (GOE) [19] since the law of the T_{ij}'s is not Gaussian and they are not all independent. However, it shares with GOE a general property of orthogonal ensembles, namely the well known 'level repulsion' [19]. One can indeed perform in (3) the change of variables $(T_{ij}) \to (\lambda_1, \ldots, \lambda_d, \theta_{1,2}, \ldots, \theta_{d-1,d})$, where the λ_i's are the d eigenvalues of T and the θ_{ij} are the $d(d-1)/2$ angles involved in the rotation diagonalizing T. A Jacobian will then show up in the expression of $P^{\circ}(T)$, namely

$$\frac{\partial T_{1,1} \ldots \partial T_{d,d}\, \partial T_{1,2} \ldots \partial T_{d-1,d}}{\partial \lambda_1 \ldots \partial \lambda_d\, \partial \theta_{1,2} \ldots \partial \theta_{d-1,d}}$$

which is a polynomial of degree $d(d-1)/2$ in $\lambda_1, \ldots, \lambda_d$ since the derivatives with respect to $\lambda_1, \ldots, \lambda_d$ yield d columns involving no λ_i's, and the remaining columns are linear in the latter. Now it is clear that this polynomial will vanish whenever two of the eigenvalues are equal, because in that case the change of variable is ambiguous, only defined up to a rotation in the (at least) two-dimensional corresponding eigenspace; hence the Jacobian is just

$$\prod_{i<j} (\lambda_i - \lambda_j) J'(\theta_{1,2}, \ldots, \theta_{d-1,d})$$

where J' is in general a complicated function. (The prefactor is what is generally referred to as level repulsion. Let us emphasize again the fact that such a prefactor occurs in the distribution of the eigenvalues of any orthogonal ensemble of random matrices; as a consequence, it would also be present in the law of the inertial tensor of self-avoiding walks, Levy flights, etc.) In our case however the form of J' is irrelevant: since the T's form an orthogonal ensemble, there is in the integrand no further dependence on the angles besides J'. Hence the angular integration reduces to a numerical factor which may be absorbed into the constant N, finally yielding the law of the eigenvalues:

$$P^{\circ}(\lambda_1, \ldots, \lambda_d) = N \prod_{i<j} |\lambda_i - \lambda_j| \int \mathrm{d}^d r_{\mathrm{A}} \mathrm{d}^d r_{\mathrm{B}} \\ \times \int_{r(0)=r_A}^{r(t)=r_B} \mathcal{D}r\, \mathrm{e}^{-\int_0^t \dot{r}^2(\tau)\mathrm{d}\tau} \prod_{k\leqslant l} \delta(\lambda_k \delta_{kl} - f_{kl}^{(1)}) \delta^d(g^{(0)}) \tag{4}$$

($f_{ij}^{(1)}$ is obtained by setting $r_{\mathrm{G}} = 0$, $T = 0$ in f_{ij} (2)). The identity $\delta(x) = 1/2\pi \int_{-\infty}^{+\infty} \mathrm{d}\alpha\, \mathrm{e}^{\mathrm{i}\alpha x}$ enables one to rewrite (4) in terms of the evolution operator in a harmonic potential:

$$P^{\circ}(\lambda_1, \ldots, \lambda_d) = N \prod_{i<j} |\lambda_i - \lambda_j| \int [\mathrm{d}\mu] \mathrm{d}^d r_{\mathrm{A}} \mathrm{d}^d r_{\mathrm{B}} \\ \times \int_{r(0)=r_A}^{r(t)=r_B} \mathcal{D}r\, \mathrm{e}^{-\int_0^t \dot{r}^2(\tau)\mathrm{d}\tau + \mathrm{i}\,\mathrm{tr}(\mu(\lambda - F^{(1)}))} \delta^d(g^{(0)}) \tag{5}$$

where we have introduced the (d, d) symmetric matrices

$$\lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_d) \qquad F^{(1)} = (f_{ij}^{(1)}) \qquad \mu = (\mu_{ij})$$

and $[\mathrm{d}\mu] = \prod_{i\leqslant j} \mathrm{d}\mu_{ij}$. We now proceed to diagonalize the matrix μ by means of a rotation R; a Jacobian similar to the previous one appears:

$$[\mathrm{d}\mu] = \left(\prod_i dw_i \right) \left(\prod_{j<k} |w_j - w_k| \right) [\mathrm{d}\theta]\, \left| J'[\theta] \right| \tag{6}$$

where $w_1, \ldots, w_d$ denote the eigenvalues of μ and $[\theta]$ stands for $\{\theta_{1,2}, \ldots, \theta_{d-1,d}\}$. The Gaussian functional integration can now be performed, yielding (see for instance [20])

$$P^{\mathrm{o}}(\lambda_1, \ldots, \lambda_d) = N \prod_{i<j} |\lambda_i - \lambda_j| \\ \times \int \left(\prod_k dw_k F^{\mathrm{o}}(w_k) \right) \left(\prod_{l<m} |w_l - w_m| \right) [d\theta] |J'[\theta]| \mathrm{e}^{\mathrm{i}\,\mathrm{tr}(R[\theta] W R[\theta]^{-1}\lambda)} \tag{7}$$

where $W = \mathrm{diag}(w_1, \ldots, w_d)$ and

$$F^{\mathrm{o}}(w) = \left(\frac{(\mathrm{i}wt)^{\frac{1}{2}}}{\sinh[(\mathrm{i}wt)^{\frac{1}{2}}]} \right)^{\frac{1}{2}} . \tag{8}$$

The function F^{o} is the result of the integration over the endpoints r_{A} and r_{B}. If, on the other hand, one imposes (before integration) the further requirement $r_A = r_B$, one straightforwardly obtains the distribution P^{c} of the moments of inertia of closed two-dimensional Brownian paths; this distribution stems from (7) through the replacement of F^{o} with

$$F^{\mathrm{c}}(w) = \frac{(\mathrm{i}wt)^{\frac{1}{2}}}{\sinh\left[\frac{1}{2}(\mathrm{i}wt)^{\frac{1}{2}}\right]} = \left[F^{\mathrm{o}}\left(\frac{w}{4}\right) \right]^2 \tag{9}$$

(incorporating irrelevant numerical factors into N as previously). We shall henceforth denote by F either one of the two functions F^{o} or F^{c}, and by P either one of P^{o} or P^{c}.

Even though the general problem has now been much simplified since the non-polynomial dependence of the eigenvalues upon the matrix elements has been removed, it is still no easy matter to integrate (7) for an arbitrary value of d. To illustrate this, we now turn (briefly) to the three-dimensional case, before describing in more detail the two-dimensional one. Let $d = 3$ and let us write R by means of the Euler angles b, a, c:

$$R = R_3(c) R_2(a) R_3(b) .$$

One gets, after some rather tedious algebra,

$$\mathrm{tr}(RWR^{-1}\lambda) = 3(\alpha\alpha' + \beta\beta'(3\cos^2 a - 1) + (\beta\gamma' \cos 2b + \gamma\beta' \cos 2c)\sin^2 a) \\ + \gamma\gamma'(\cos 2b \cos 2c(1 + \cos^2 a) - 2 \sin 2b \sin 2c \cos a) \tag{10}$$

$$J'(a, b, c) = \sin a \tag{11}$$

where $\alpha = (w_1 + w_2 + w_3)/3$, $\beta = (2w_3 - w_1 - w_2)/6$, $\gamma = (w_1 - w_2)/2$ and α', β', γ' are obtained from α, β, γ by replacing w_i's with λ_i's. Even though the integration over one angle may be performed, it seems difficult to go much further in three dimensions.

In $d = 2$, the situation is more favourable (define $\theta = \theta_{1,2}$):

$$\mathrm{tr}(RWR^{-1}\lambda) = \tfrac{1}{2}(w_1 + w_2)(\lambda_1 + \lambda_2) + \tfrac{1}{2}\cos 2\theta (w_1 - w_2)(\lambda_1 - \lambda_2) \tag{12}$$

$$J'(\theta) = 1 . \tag{13}$$

The spectral distribution then reads

$$P(\lambda_1, \lambda_2) = N|\lambda_1 - \lambda_2| \iint \mathrm{d}w_1 \mathrm{d}w_2 F(w_1) F(w_2) |w_1 - w_2|$$
$$\times J_0\left(\tfrac{1}{2}(w_1 - w_2)(\lambda_1 - \lambda_2)\right) \mathrm{e}^{\frac{1}{2}\mathrm{i}(w_1+w_2)(\lambda_1+\lambda_2)} \tag{14}$$

where J_0 is the Bessel function of order 0. Many interesting results can be obtained from this general expression. (Let us note that an expression for $P^{\mathrm{c}}(\lambda_1, \lambda_2)$ has already been obtained in [16].) Since our main purpose is to provide a description of the shape of the paths, we are particularly interested in the distribution of the difference of the eigenvalues; more precisely, consider the law (still denoted by P) of the random variable $x = |\lambda_1 - \lambda_2|/t$, which is a universal function since T scales as t. Integrating over $\lambda_1 + \lambda_2$, one readily gets, for $x \geqslant 0$,

$$P^{\mathrm{o}}(x) = Nx \int_0^\infty \mathrm{d}y\, y^{\frac{3}{2}} J_0(yx)\left(\cosh\sqrt{2y} - \cos\sqrt{2y}\right)^{-\frac{1}{2}} \tag{15}$$

and

$$P^{\mathrm{c}}(x) = Nx \int_0^\infty \mathrm{d}y\, y^2 J_0(yx)\left(\cosh\sqrt{\frac{y}{2}} - \cos\sqrt{\frac{y}{2}}\right)^{-1} \tag{16}$$

clearly $P(x) = 0$ for $x < 0$.

The asymptotic behaviours of $P(x)$ are readily obtained: in particular, the tail of the distribution is seen to be an exponential. When $x \to \infty$, the integral is dominated by the small-y part:

$$P^{\mathrm{o}}(x) \sim x \int_0^\infty \mathrm{d}y\, y\, J_0(yx)(90 + y^2)^{-\frac{1}{2}} = \mathrm{e}^{-\sqrt{90}x} \tag{17}$$

with an analogous form for P^{c}, whereas when $x \to 0$ $P(x) \sim x$. (These asymptotic forms of P^{o} and P^{c} may be viewed as the probability laws of very elongated paths ($x \to \infty$) and quasi-spherical ones ($x \to 0$).)

We present in figure 1 a comparison of a numerical calculation of (15) (full curve, representing an interpolation of 40 calculated points) and a computer simulation of open random walks on a square lattice (full circles). We generated 10^6 walks of 1000 steps each. The agreement with (15) is quite satisfying.

The profile of this distribution already provides meaningful information about the shapes of the paths; though $P^{\mathrm{o}}(x)$ decreases exponentially when x becomes large, important deviations from sphericity are allowed for: in fact it is seen that the small values of the difference $|\lambda_1 - \lambda_2|$ are suppressed in favour of values of significant magnitude (i.e. $x \simeq 0.1$, or $|\lambda_1 - \lambda_2| \simeq R^2$, since $\langle R^2 \rangle^{\mathrm{o}} = t/6$ [21]). This is due to the level repulsion exhibited by $P^{\mathrm{o}}(x)$ for small x. One could also check that, even though the same level repulsion exists in the closed-path case, the distribution is then thinner, leading to the expected conclusion that closed paths are more spherical than open ones.

Though we can gain information on the typical shape of the paths just by looking at the distribution of the difference of the eigenvalues, it would still be interesting to define a numerical quantity characterizing the average elongation. A good candidate in this respect is the ratio of the average *distinct* moments of inertia $\langle\lambda_+\rangle/\langle\lambda_-\rangle$, where λ_+ (λ_-) denotes the largest (smallest) eigenvalue of the path. The Heaviside distribution

$$\theta(x) = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{+\infty} \mathrm{d}u \frac{\mathrm{e}^{\mathrm{i}ux}}{u - \mathrm{i}\epsilon}$$

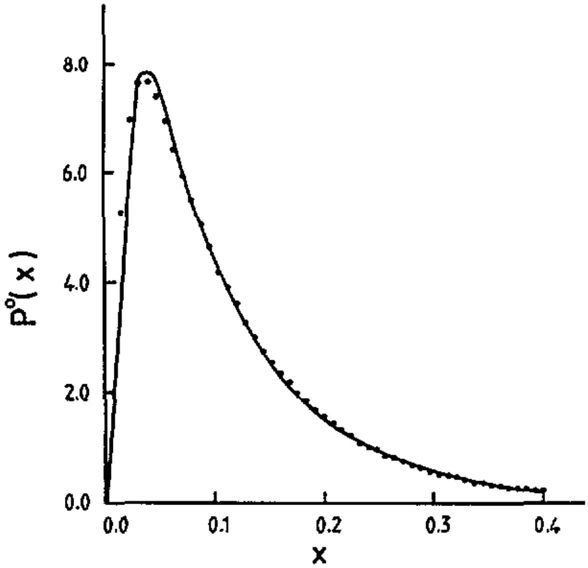

Figure 1. Computer simulation of open random walks on a two-dimentional square lattice. The probability distribution $P^{\mathrm{o}}(x)$ is (full circles) as a function of the scaling vasriable $x = |\lambda_1 - \lambda_2|/t$, where $t = 1000$ is the number of a walk and λ_1, λ_2 are the eigenvalues of the inertial tensor. The full curve represents a numerical calculation of (15).

enables us to explicitly distinguish the two eigenvalues, through the insertion of a factor $\theta(\lambda_+ - \lambda_-)$ in (14). We thus obtain the characteristic function

$$\langle \mathrm{e}^{-\mathrm{i}p_+\lambda_+ - \mathrm{i}p_-\lambda_-} \rangle = N(\partial_{p_+} - \partial_{p_-}) \int \frac{\mathrm{d}u \mathrm{d}q}{u - \mathrm{i}\epsilon} F(w_+)F(w_-) \tag{18}$$

where $w_\pm$ denote the eigenvalues of the matrix

$$\begin{pmatrix} p_+ - u & q/2 \\ q/2 & p_- + u \end{pmatrix} \tag{19}$$

whence we deduce the desired ratio

$$\frac{\langle \lambda_+ - \lambda_- \rangle}{\langle \lambda_+ + \lambda_- \rangle} = \left. \frac{\partial_{p'}^2 G(p, p')}{\partial_p \partial_{p'} G(p, p')} \right|_{p=p'=0} \tag{20}$$

where

$$\begin{aligned} &p = p_+ + p_- \quad p' = p_+ - p_- \qquad G(p, p') = \int \frac{\mathrm{d}u\, \mathrm{d}q}{u - \mathrm{i}\epsilon} F(w'_+)F(w'_-) \\ &w'_\pm = p \pm \sqrt{(p' - u)^2 + q^2}\,. \end{aligned} \tag{21}$$

Using

$$\frac{1}{u - \mathrm{i}\epsilon} = PP\left(\frac{1}{u}\right) + \mathrm{i}\pi\delta(u)$$

we get (after some algebra) for open paths

$$\frac{\langle\lambda_+ - \lambda_-\rangle}{\langle\lambda_+ + \lambda_-\rangle} = \frac{4\pi\alpha I^{\mathrm{o}}}{2\pi\partial_p(\sqrt{p}/\sinh(\alpha\sqrt{p}))|_{p=0}} = -12I^{\mathrm{o}} \tag{22}$$

where $\alpha = \mathrm{e}^{\mathrm{i}\pi/4}$ and

$$I^{\mathrm{o}} = \int_0^\infty \frac{\mathrm{d}x}{x^2}\partial_x\left(\frac{x}{\sqrt{\cosh x - \cos x}}\right) \simeq -0.0555 \tag{23}$$

leading to the value

$$\frac{\langle\lambda_+\rangle}{\langle\lambda_-\rangle} \simeq 4.99 \tag{24}$$

which is in perfect agreement with the one obtained from our computer simulations. In the case of closed paths, one has

$$\frac{\langle\lambda_+ - \lambda_-\rangle}{\langle\lambda_+ + \lambda_-\rangle} = \frac{4\pi I^{\mathrm{c}}}{2\pi\partial_p(\sqrt{p}/\sinh(\alpha\sqrt{p}))^2|_{p=0}} = -6I^{\mathrm{c}} \tag{25}$$

where

$$I^{\mathrm{c}} = \int_0^\infty \frac{\mathrm{d}x}{x^2}\partial_x\left(\frac{x^2}{\cosh x - \cos x}\right) \tag{26}$$

$$= -2\ln\left(\prod_{p=0}^{\infty}(1+\mathrm{e}^{-(2p+1)\pi})\right) \tag{27}$$

$$= -\frac{1}{2}\left(\ln 2 - \frac{\pi}{6}\right)$$
$$\simeq -0.0848\,.$$

Thus

$$\frac{\langle\lambda_+\rangle}{\langle\lambda_-\rangle} \simeq 3.07 \tag{28}$$

which is smaller than (24), as expected.

As we have already mentioned, another quantity of interest is the square radius of gyration; the latter is obviously a measure of the *size* of a walk, and conveys no information whatsoever about its shape. It is nevertheless interesting to compare its distribution Q with those of the differences of eigenvalues, as shown in figure 1. All the more so as it may be expressed quite generally under a relatively simple form. Here we consider the random variable

$$z = \frac{1}{t}\sum_{i=1}^{d}\lambda_i \tag{29}$$

d now being arbitrary, and compute the Laplace transform $\Gamma_d(p) = \langle\mathrm{e}^{-pz}\rangle$. One has

$$\Gamma_d^{\mathrm{o}}(p) = N\int \mathrm{d}^d r_{\mathrm{A}}\mathrm{d}^d r_{\mathrm{B}}\mathrm{d}^d k \int_{r(0)=r_{\mathrm{A}}}^{r(t)=r_{\mathrm{B}}} \mathcal{D}r\, \mathrm{e}^{-\int_0^1 \mathrm{d}\tau(\dot{r}^2+pr^2+\mathrm{i}k\cdot r)} = \left(\frac{\sqrt{p}}{\sinh\sqrt{p}}\right)^{d/2} \tag{30}$$

and

$$\Gamma_d^{\rm c}(p) = \left(\frac{\sqrt{p}}{2} / \sinh \frac{\sqrt{p}}{2}\right)^d = \Gamma_{2d}^{\rm o}\left(\frac{p}{4}\right) . \tag{31}$$

It remains to compute the inverse transforms of (30))and (31):

$$Q_d(z) = \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}p \, \Gamma_d(p) \mathrm{e}^{pz} . \tag{32}$$

One has, by virtue of (31),

$$Q_d^{\rm c}(z) = 4 \, Q_{2d}^{\rm o}(4z) \tag{33}$$

(using this result, we can immediatlly check the long known [22, 21] formula $\langle R^2\rangle^{\rm c} = 1/2\langle R^2\rangle^{\rm o}$); moreover $Q_d^{\rm o}$ can be expressed as a series for even d, and $Q_d^{\rm c}$ for any d. For instance, we get (setting $z' = \pi^2 z$ and still denoting by Q the law of z')

$$Q_2^{\rm o}(z') = 2\,\theta(z') \sum_{n=1}^{\infty} (-1)^{n+1} n^2 \mathrm{e}^{-n^2 z'} \tag{34}$$

$$Q_2^{\rm c}(z') = 8\,\theta(z') \sum_{n=1}^{\infty} n^2 (8n^2 z' - 3) \mathrm{e}^{-4n^2 z'} . \tag{35}$$

(An analytical expression for $Q_2^{\rm o}(z)$ was first obtained in [7]; see also [25] for the derivation of the equivalent of (35) in another context.) As was already noticed [5], Q exhibits a non-analytical behaviour near the origin; for instance,

$$Q_2^{\rm o}(z') \underset{z' \to 0}{\sim} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}p \, \mathrm{e}^{-\pi\sqrt{p} + pz'} \sim \mathrm{e}^{-\pi^2/4z'} . \tag{36}$$

Rescaling z to $z'' = z/d$, we can investigate the limit $d \to \infty$:

$$\langle \mathrm{e}^{-pz''} \rangle = \left(\sqrt{\frac{p}{d}} \Big/ \sinh \sqrt{\frac{p}{d}}\right)^{d/2} \to \mathrm{e}^{-p/12} \tag{37}$$

yielding

$$Q_d^{\rm o}(z'') \underset{d \to \infty}{\to} \delta\left(z'' - \tfrac{1}{12}\right) \tag{38}$$

while

$$Q_d^{\rm c}(z'') \underset{d \to \infty}{\to} \delta\left(z'' - \tfrac{1}{24}\right) . \tag{39}$$

This shows that the variable z'' becomes more and more peaked about its average value when the number of dimensions increases. In fact, all meaningful quantities are expected to

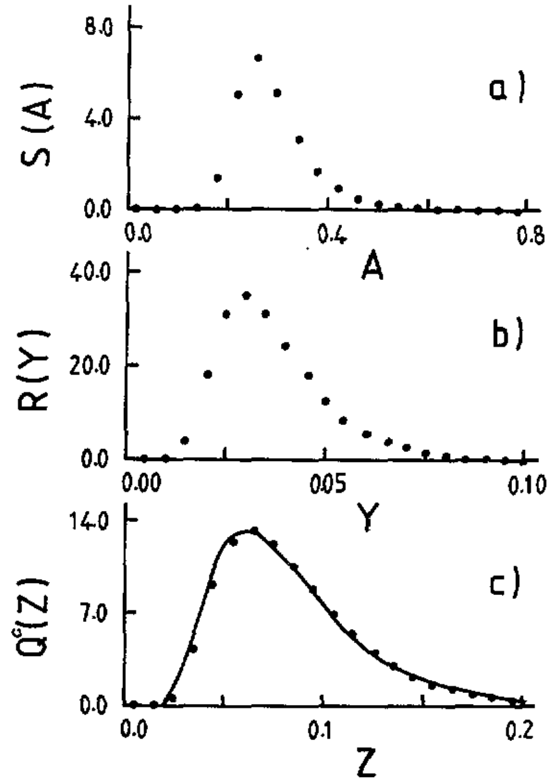

Figure 2. Computer simulation of closed random walks on a two-dimensional square lattice. The probability distributions (*a*) $S(A)$, (*b*) $R(y)$, (*c*) $Q^c(z)$ are plotted (full circles) as functions of the scaling variables $A = \mathcal{A}/t$, $y = \sqrt{\lambda_1\lambda_2}/t$ and $z = |\lambda_1 + \lambda_2|/t$, where $t = 5000$ is the number of steps of a walk, $\mathcal{A}$ is the arithmetic area enclosed by a walk and λ_1, λ_2 are the eigenvalues of the inertial tensor. The full curve represents a numerical calculation of (35) ($z' = \pi^2 z$). The area $\mathcal{A}$ appears to be better correlated with y than with z. See text for more detailed explanations.

have a simple limit when d becomes large; for instance, the values of the average moments of inertia in large dimensions have been computed as a series in $1/d$ [17].

We finally examine the issue of the arithmetic area $\mathcal{A}$ enclosed by the closed planar Brownian path. The latter is very difficult to evaluate and analytical derivation of its probability law is still an open problem [23]. However, it is natural to expect that the area enclosed has something to do with the moments of inertia λ_1 and λ_2, and there are indeed numerical indications of this. Let us investigate the correlations between $A = \mathcal{A}/t$ and the variables $z = (\lambda_1 + \lambda_2)/t$ and $y = \sqrt{\lambda_1\lambda_2}/t$; clearly A would be exactly proportional to z if the path uniformly filled a disk, and to y if it uniformly filled the interior of an ellipse. The distributions of A, z and y are displayed in figure 2. The law of the area is obtained by generating 200 000 walks of 5000 steps on a square lattice and computing A for each walk by means of a site-bond percolation algorithm [24]; the other laws plotted are also simulation results (full circles), and that of z is seen to coincide with the analytical expression (35) (full curve). Comparing the three distributions, one notices that the law of A is more similar to that of y than to that of z, as could be expected. The correlation coefficients are

$$C_{Az} \simeq 0.76 \qquad C_{Ay} \simeq 0.87\,. \tag{40}$$

There is of course no linear relation between those variables, but it is noteworthy that the arithmetic area, which takes into account every detail of the 'tortured' history of the Brownian path, should be so well approximated by the area enclosed by the 'inertial ellipse', a smoothed (and far simpler to evaluate) quantity. This might prove an interesting remark in the light of some recent attempts to use closed pressurized Brownian paths as a model for two-dimensional vesicles [25–28], as the enclosed area, being the conjugate variable to osmotic pressure, plays an important role in such a model.

After the completion of this paper, our attention was brought to reference [27], in which the form of $P_2^c(\lambda_1, \lambda_2)$ (expression (14) with $F = F^c$ taken from (9)) has been independently derived.

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